UGC – Minor Research Project in Mathematics
Executive Summary of the Final Report

on

A STUDY ON LINEAR ALGEBRAIC ASPECTS OF SEMIRINGS OF MATRICES

Submitted by

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A STUDY ON LINEAR ALGEBRAIC ASPECTS OF SEMIRINGS OF MATRICES

Introduction

The set $M_n(X)$ of all $n \times n$ matrices over an algebra X has been studied for various choices of X. The usual matrix theory considers $M_n(X)$ where X is the set of all real numbers or more generally a field. When K is a field, the set $M_n(X)$ of all $n \times n$ matrices over K has the natural structures of a ring and a vector space over K. Various aspects of $M_n(X)$ have been studied in Linear Algebra.

Matrices over rings, matrices over lattices, special matrices over semigroups etc. are much studied and these studies resulted in a number of different structures on the set of such matrices. The appropriate setting for studying matrices over lattices is semirings. The notion of semiring was introduced by H. S. Vandiver in the year 1934 in his pioneer paper entitled "Note on a simple type of algebra in which the cancellation law of addition does not hold", published in Bulletin of American Mathematical Society. He defined a semiring or associative algebra as a set of elements forming a semigroup under addition, a semigroup under multiplication, and in which the right and left distributive laws hold. The natural examples of semirings are the set of all natural numbers under usual addition and multiplication, a distributive lattice when sum is l.u.b. and product is g.l.b. etc.

The Principal Investigator in his doctoral thesis discussed some linear algebra relations in $M_n(D)$ where D is a distributive lattice. The concept of semimodule was considered there. In this project the consistency of the linear equations whose coefficients are from a distributive lattice D has been checked and proved that the set of all solutions of a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ on D is a d-subspace of D_n.

Chapter 1 : Preliminaries

In the first chapter we discuss basic concepts and some results in semigroup theory and lattice theory.

Chapter 2 : Semirings of Matrices over Lattices

In this chapter we introduce semirings and the semiring of matrices. We describe various concepts related to semirings and some of its properties. Here we proved that $M_n(D)$ is a semiring of matrices over the distributive lattice D. We may consider semiring of matrices over lattices in general. But it turns out that they are all matrices over distributive lattices. Hence we proved the result : Let $(L, +, \cdot)$ be a lattice, then $M_n(L)$ is a semiring if and only if L is a distributive lattice.

Chapter 3 : Linear Algebra over $M_n(D)$

In the third chapter we discuss some linear algebra relations in $M_n(D)$. We consider n-tuples of elements of D as vectors and analyse the relations among row vectors and column vectors of a matrix in $M_n(D)$. Behaviour of set of solutions of linear equations and invertibility of matrices will be considered.

In the first section we consider distributive lattice D with greatest element 1 and minimal element 0. We give an algebraic structure for D^n and regard elements of D^n as D-vectors. All semirings $(S, +, \cdot)$ are assumed to have $(S, +)$ as a semilattice and containing $1 \in S$. We can consider the set of all $m \times n$ matrices $M_{m,n}(S)$ as a semimodule over S. We also discuss some nice examples of semimodule, d-subspace in this section.

In the next section we consider conditions for consistency of linear equations in D in terms of the semimodule D^n. The set of all solutions will be related to certain d-subspaces. Here D is a distributive lattice with least element 0 and greatest element 1. We begin by developing consistency conditions for a linear equation, in n variables. A solution of the linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ is a vector $(y_1, y_2, \ldots, y_n) \in D^n$ such that $a_1y_1 + a_2y_2 + \cdots + a_ny_n = d$. We proved the following results in this session.

(1) A linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ where $a_i, d \in D$ is consistent if and only if $d \leq a_1 + a_2 + \cdots + a_n$.

(2) The set of all solutions of a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$ over D is a d-subspace of D^n.

(3) The set of all solutions of a homogeneous system $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ is a subspace of D^n.

For checking the consistency of a linear system we have proved the three lemmas in the second section and using these lemmas we obtained the following result.
One sufficient condition for the system

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \alpha$$
$$b_1x_1 + b_2x_2 + \cdots + b_nx_n = \beta$$

to be consistent is that $a_1 + a_2 + \cdots + a_n = \alpha$ and $b_1 + b_2 + \cdots + b_n = \beta$.

Finally we conclude this section by proving a sufficient condition for the consistency of a system as follows:
Consider a system of m linear equations

$$\begin{aligned}
a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n} &= \alpha_1 \\
a_{21}x_{21} + a_{22}x_{22} + \cdots + a_{2n}x_{2n} &= \alpha_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_{m1} + a_{m2}x_{m2} + \cdots + a_{mn}x_{mn} &= \alpha_m
\end{aligned}$$

A sufficient condition for the consistency of the above system is $\sum_i a_{ij} = \alpha_i$ for every i. Further every $y = (y_1, y_2, \ldots, y_n)$ such that $y_j \geq \sum_i a_{ij}$ is a solution of the system.

The project work is concluded by establishing the following characterization of invertible matrices in $M_n(D)$ in the last section of this chapter.

A matrix $A \in M_n(D)$ is invertible if and only if A is orthogonal that is $A^{-1} = A^T$.